GLASNIK MATEMATIČKI
Vol. 39(59)(2004), 207 – 211

COPRODUCTS FOR CLIFFORD ALGEBRAS

Pavle Pandžić
University of Zagreb, Croatia

Abstract. We define a family of graded coproducts for Clifford algebras over finite dimensional real or complex vector spaces and study their basic properties.

1. Introduction

Let V be a finite-dimensional real or complex vector space. Let B be a symmetric bilinear form on V. The Clifford algebra $C(V)$ for the form B is defined as the quotient of the tensor algebra $T(V)$ by the ideal generated by elements of the form

$$v \otimes v + B(v,v), \qquad v \in V. \tag{1}$$

Note that $C(V)$ does not inherit the natural $\mathbb{Z}$-grading of $T(V)$, but it does inherit the corresponding $\mathbb{Z}_2$-grading, as the relations are even. So we can also view $C(V)$ as a superalgebra. Also, $C(V)$ inherits the natural filtration from $T(V)$ and the corresponding graded algebra is the exterior algebra $\bigwedge(V)$. There is a canonical 'antisymmetrization' (or 'quantization') map due to Chevalley [Ch] from $\bigwedge(V)$ into $C(V)$, which is a linear inverse of the natural projection. In particular, the dimension of $C(V)$ is $2^{\dim V}$.

The structure theory and classification for Clifford algebras and their modules (over $\mathbb{R}$ or $\mathbb{C}$ if V is real, or just over $\mathbb{C}$ if V is complex) are well known and easy to describe. It turns out that $C(V)$ is either a matrix algebra or a sum of two matrix algebras. Consequently the modules form a semisimple category with either one or two irreducible objects; these are called the spin

2000 *Mathematics Subject Classification.* 11E88, 15A66, 16W30.

Key words and phrases. Clifford algebra, superalgebra, graded coproduct, coalgebra.

This article was written during author's stay at RIMS, Kyoto University, Japan, as a fellow of The Matsumae International Foundation.

modules. One can equally easily describe the $\mathbb{Z}_2$-graded modules. A good account for all this is in [LM]; the results are mostly from [ABS] and [Ch].

In spite of (or perhaps because of) this simplicity, Clifford algebras and modules play an enormously important role in geometry, analysis and physics. This is well illustrated in [LM]. For example, the definitions of the Dirac operator and its various analogues require the Clifford algebra setting, so in fact the celebrated Atiyah-Singer index theorem uses Clifford algebras in an essential way. In physics, Clifford formalism describes the so called fermions. Various aspects, applications and generalizations of Clifford algebras and modules are being studied intensively; suffice it to say there is a whole journal devoted to these developments, Advances in Applied Clifford Algebras.

There have also been important applications to representation theory of real reductive groups. Among many such results, let us mention the construction of the discrete series representations [P], [AS], and some nice recent results of Kostant [K] of a different flavor. Also, in [HP1], [HPR] we have obtained some results using an algebraic version of the Dirac operator due to Vogan. An introduction to this setting can be found in [HP2], which also briefly mentions a special case of the construction of coproducts given below.

While in [HPR] we are investigating the role of Dirac operators in understanding the nilpotent Lie algebra cohomology of Harish-Chandra modules, Dirac operators also appear in studying the $(\mathfrak{g}, K)$-cohomology of (unitary) Harish-Chandra modules; see [W], [BW]. In trying to understand this last relationship, one comes accross identities like

$$C(V) = S \otimes S; \tag{2}$$

here V is an even dimensional complex vector space and S is the unique spin module for $C(V)$. While the two spaces in the identity (2) obviously have the same dimension, the identity itself is ambiguous, since we have not said what actions are considered, and what map should identify the two sides. For example, (2) can be interpreted as an equality of $C(V) \otimes C(V)$-modules, where the action of the second $C(V)$ on $C(V)$ is given by the right multiplication twisted by the transpose map $\tau : C(V) \to C(V)$ which is the unique antiautomorphism given as id on $V \subset C(V)$. See [LM], Proposition 5.18.

Note that the above defined τ can also be used to define the dual module S^*, which by uniqueness has to be isomorphic to S.

Getting back to general V, one can ask if we can define an interior tensor product of $C(V)$-modules, i.e., if we can define a natural $C(V)$-module structure on $S_1 \otimes S_2$ for $C(V)$-modules S_1 and S_2. This is naturally related to the existence of coproducts, i.e., algebra homomorphisms from $C(V)$ into $C(V) \otimes C(V)$; indeed, if c is any such homomorphism, we can use the obvious $C(V) \otimes C(V)$-action on $S_1 \otimes S_2$ and pull it back to $C(V)$ via c. [LM] offers two easy answers (just above Proposition 5.18); we can have all the action on the first factor only, or on the second factor only. These correspond to

c being one of the obvious embeddings. Each of these two actions gives an interpretation of (2).

The purpose of this note is to exhibit a whole family of *graded* coproducts, and thus tensor structures on the category of graded modules. These coproducts are given by

$$c_t(v) = \cos t\, v \otimes 1 + \sin t\, 1 \otimes v, \qquad v \in V, \tag{3}$$

for a real parameter t. In particular, for $t = 0$ respectively $t = \frac{\pi}{2}$ we are getting the two embeddings as above; for $t \in (0, \frac{\pi}{2})$, c_t can be thought of as interpolations between these two choices. The most symmetric choice, $t = \frac{\pi}{4}$, makes both factors contribute equally to the tensor product action.

Besides the possible use in calculations like the ones mentioned above related to $(\mathfrak{g}, K)$-cohomology, the obtained "pseudo-Hopf" structure seems interesting in its own right. For example, the obtained coproducts are coassociative only for $t = 0$ or $t = \frac{\pi}{2}$; on the other hand the only cocommutative one corresponds to $t = \frac{\pi}{4}$. There is no counit, as $C(V)$ does not act on $\mathbb{C}$; however, there is at least one candidate for something like an "antipode" - the above mentioned τ.

In Section 2 we check that the coproducts given by (3) are indeed well defined, and that they have the above mentioned properties. The reader not familiar with the language of coalgebras can consult the book [M].

2. Coalgebra structure on $C(V)$

It is clear from the definition of $C(V)$ that it satisfies the following universal property: for any linear map ϕ from V into an associative algebra A with unit, such that the relations (1) are satisfied, i.e., $\phi(v)^2 = -B(v, v)$, for all $v \in V$, there is a unique algebra homomorphism $\Phi : C(V) \to A$ extending ϕ.

Thus, to check that (3) defines an algebra homomorphism c_t from $C(V)$ into the graded tensor product $C(V)\hat{\otimes}C(V)$ for each t, it is enough to note that

$$\begin{aligned}
c_t(v)^2 &= (\cos t\, v \otimes 1 + \sin t\, 1 \otimes v)(\cos t\, v \otimes 1 + \sin t\, 1 \otimes v) \\
&= \cos^2 t\, v^2 \otimes 1 + \cos t \sin t (v \otimes v - v \otimes v) + \sin^2 t\, 1 \otimes v^2 \\
&= (\cos^2 t + \sin^2 t) B(v, v) \cdot 1 \otimes 1 = B(v, v).
\end{aligned}$$

Note how it was essential to consider the graded tensor product, in order to have $(1 \otimes v)(v \otimes 1) = -v \otimes v$.

Of course, we need not consider all $t \in \mathbb{R}$, the relevant t are in $[0, 2\pi)$. Moreover, changing sign in one or both factors can be interpreted as applying the unique automorphism α of $C(V)$ given as $-\operatorname{id}$ on V, and hence is also redundant. This means we can restrict the parameter to $t \in [0, \frac{\pi}{2}]$. We have proved

Theorem 2.1. *For any* $t \in [0, \frac{\pi}{2}]$, *(3) defines a coproduct* c_t *on* $C(V)$ *compatible with the superalgebra structure, i.e., a superalgebra homomorphism from* $C(V)$ *into the* $\mathbb{Z}_2$*-graded tensor product* $C(V)\hat{\otimes}C(V)$.

Let us now examine whether c_t is coassociative or cocommutative. Coassociativity of a (graded) coproduct c means that the following digram commutes:

$$\begin{array}{ccc} C(V) & \xrightarrow{c} & C(V)\hat{\otimes}C(V) \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c\otimes \mathrm{id}} \\ C(V)\hat{\otimes}C(V) & \xrightarrow{\mathrm{id}\otimes c} & C(V)\hat{\otimes}C(V)\hat{\otimes}C(V) \end{array}$$

For $c = c_t$, the first row evaluated at $v \in V$ gives

$$\cos^2 t\, v \otimes 1 \otimes 1 + \cos t \sin t\, 1 \otimes v \otimes 1 + \sin t\, 1 \otimes 1 \otimes v,$$

while the second row evaluated at $v \in V$ gives

$$\cos t\, v \otimes 1 \otimes 1 + \cos t \sin t\, 1 \otimes v \otimes 1 + \sin^2 t\, 1 \otimes 1 \otimes v.$$

The two expressions are equal if and only if $t = 0$ or $t = \frac{\pi}{2}$.

Cocommutativity (in the graded sense) of a coproduct c means that the following diagram commutes:

$$\begin{array}{ccc} C(V) & \xrightarrow{c} & C(V)\hat{\otimes}C(V) \\ {\scriptstyle =}\downarrow & & {\scriptstyle \gamma}\downarrow \\ C(V) & \xrightarrow{c} & C(V)\hat{\otimes}C(V) \end{array}$$

where γ is the *graded twist map*: $a \otimes b \mapsto (-1)^{\deg a \deg b} b \otimes a$. Since 1 is even, $\gamma(v \otimes 1) = 1 \otimes v$ and $\gamma(1 \otimes v) = v \otimes 1$; so c_t is cocommutative if and only if $\cos t = \sin t$, i.e., $t = \frac{\pi}{4}$. We have proved

Proposition 2.2. *The coproduct* c_t *is coassociative if and only if* $t = 0$ *or* $t = \frac{\pi}{2}$, *and cocommutative if and only if* $t = \frac{\pi}{4}$.

To finish, let us mention that the theory of coalgebras and comodules has proved powerful and useful in contexts like algebraic group actions, quantum groups or homotopical algebra. Therefore it could be of interest to investigate $C(V)$-comodules.

References

[ABS] M.F. Atiyah, R. Bott and A. Shapiro, *Clifford modules*, Topology **3**, Suppl. 1 (1964), 3–38.

[AS] M. Atiyah and W. Schmid, *A geometric construction of the discrete series for semisimple Lie groups*, Invent. Math. **42** (1977), 1–62.

[BW] A. Borel and N. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, Second edition. Mathematical Surveys and Monographs, 67. American Mathematical Society, Providence, RI, 2000.

[Ch] C. Chevalley, *The algebraic theory of spinors*, Columbia University Press, 1954.

[HP1] J.-S. Huang and P. Pandžić, *Dirac cohomology, unitary representations and a proof of a conjecture of Vogan*, J. Amer. Math. Soc. **15** (2002), 185–202.

[HP2] J.-S. Huang and P. Pandžić, *Dirac operators in representation theory*, in Representations of Real and p-Adic Groups, Lecture Notes Series vol. 2 (Tan, E.-C., Zhu, C.-B., eds.) Singapore University Press and World Scientific Publishing, 2004, 163-219.

[HPR] J.-S. Huang, P. Pandžić and D. Renard, *Dirac operators and Lie algebra cohomology*, Preprint, 2003.

[LM] H. B. Lawson and M.-L. Michelsohn, *Spin Geometry*, Princeton Univ. Press, 1989.

[K] B. Kostant, *Clifford algebra analogue of the Hopf-Koszul-Samelson theorem, the ρ-decomposition $C(\mathfrak{g}) = End\, V_\rho \otimes C(P)$, and the $\mathfrak{g}$-module structure of $\bigwedge \mathfrak{g}$*, Adv. Math. **125** (1997), 275–350.

[M] S. Montgomery, *Hopf algebras and their actions on rings*, CBMS Regional Conference Series in Mathematics 82, American Mathematical Society, 1993.

[P] R. Parthasarathy, *Dirac operator and the discrete series*, Ann. of Math. **96** (1972), 1–30.

[W] N. R. Wallach, *Real Reductive Groups, Volume I*, Academic Press, 1988.

P. Pandžić
Department of Mathematics
University of Zagreb
Bijenička 30, PP 335, 10002 Zagreb
Croatia
E-mail: `pandzic@math.hr`

Received: 21.11.2003.